General-relativistic viscous fluids

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Mathematical and Computational Approaches for the Einstein Field Equations with Matter Fields

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Relativistic ideal fluids

A (relativistic) ideal fluid is described by the (relativistic) Euler equations

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0,$$
$$\nabla_\alpha J^\alpha = 0,$$

where $\mathcal{T}$ is the energy-momentum tensor of an ideal fluid given by

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Above, ϱ is the fluid's (energy) density, n is the baryon density, $p = p(\varrho, n)$ is the fluid's pressure, and u is the fluid's (four-)velocity, which satisfies

$$g_{\alpha\beta} u^\alpha u^\beta = -1.$$

g is the spacetime metric and ∇ the corresponding covariant derivative.

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There are, however, important situations where a theory or relativistic viscous fluids is needed.

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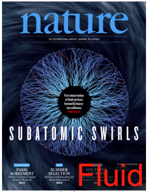

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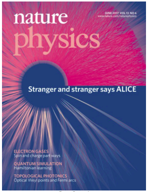

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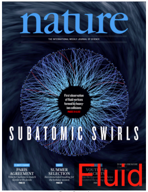

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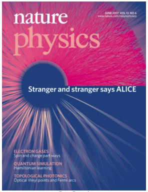

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Theory, experiments, numerical simulation, phenomenology: the QGP is a relativistic liquid with viscosity.

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Conclusion: Knudsen number $K_n \sim \ell/L$ may not be small in some cases $\Rightarrow$ viscous contributions likely to affect the gravitational wave signal.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

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First-order theory: $\pi = \pi(\varrho, u, \partial\varrho, \partial u, \dots)$ etc.

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Second-order theory: $u^\mu \nabla_\mu \pi + \cdots = 0$ etc.

The Eckart and Landau-Lifshitz theories

Starting from:

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2. Entropy production ≥ 0.

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Instability/acausality results apply to large classes of first-order theories. Difficult to construct causal and stable theories of relativistic fluids with viscosity: great deal of work trying to address the issue.

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

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$$\begin{aligned}
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where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

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$$\mathcal{J}' s = \partial\varrho + \partial u + \partial\mathcal{P} + \partial\mathcal{Q} + \partial\pi$$

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System is highly complex; e.g., $\mathcal{Q} = 0$, 22×22 system with non-diagonal principal part.

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Proof by contradiction: it does not reveal the nature of the singularity; first breakdown result for Israel-Stewart.

The BDNK theory

The BDNK theory is a first-order theory defined by (D-Bemfica-Noronha, '18, '19, '20; Kovtun, '19; Hoult-Kovtun, '20):

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

with

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Lots of terms: need them to fix the causality and instability problems of Eckart and Landau-Lifshitz. One should let the fundamental principle of causality constrain which terms are allowed in the theory rather than decide the possible terms and then try to establish causality

Theorem: Causality, stability, and LWP of the BDNK theory (D-Bemfica-Rodriguez-Shao, '19; D-Bemfica-Graber, '20; D-Bemfica-Noronha, '20)

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Theorem in fact valid with baryon current and $p = p(\varrho, n)$.

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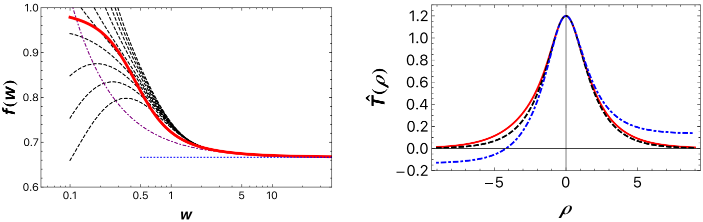

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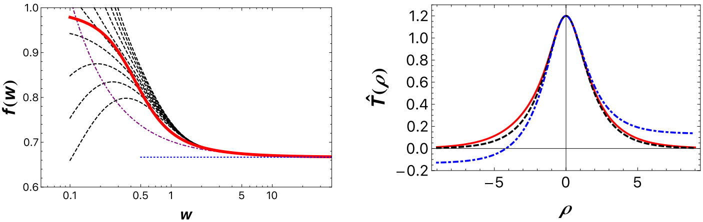

The BDNK theory has all the good features of the Israel-Stewart theory plus a good local well-posedness theory in Sobolev spaces, which is lacking for Israel-Stewart (applications to neutron star mergers).

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