General-relativistic viscous fluids

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Relativistic ideal fluids

A (relativistic) ideal fluid is described by the (relativistic) Euler equations

$$\nabla_\alpha T^\alpha_\beta = 0,$$
$$\nabla_\alpha J^\alpha = 0,$$

where $T$ is the energy-momentum tensor of an ideal fluid given by

$$T_{\alpha\beta} = (p + \rho) u_\alpha u_\beta + pg_{\alpha\beta},$$

and $J$ is the baryon current of an ideal fluid given by

$$J_\alpha = nu_\alpha.$$
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Above, \( \rho \) is the fluid’s (energy) density, \( n \) is the baryon density, \( p = p(\rho, n) \) is the fluid’s pressure, and \( u \) is the fluid’s (four-)velocity, which satisfies

\[ g_{\alpha\beta}u^\alpha u^\beta = -1. \]

\( g \) is the spacetime metric and \( \nabla \) the corresponding covariant derivative.
The need for relativistic viscous fluids

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There are, however, important situations where a theory or relativistic viscous fluids is needed.
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Theory, experiments, numerical simulation, phenomenology: the QGP is a relativistic liquid with viscosity.
Neutron star mergers

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$$\text{Knudsen number } K_n \sim \ell / L$$

$\Rightarrow$ viscous contributions likely to affect the gravitational wave signal.
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From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

\[
T_{\alpha\beta} := (\rho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + Q_{\alpha}u_\beta + Q_{\beta}u_\alpha,
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quantities as before \((u_\alpha u^\alpha = -1)\); \(\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta\).
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- **First-order:** \(R, P, Q,\) and \(\pi\) given in terms of \(\rho, u,\) and their derivatives.
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First-order theory: $\pi = \pi(\varrho, u, \partial_\varrho, \partial u, \ldots)$ etc.
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- **Second-order:** \( R, P, Q, \) and \( \pi \) are new variables treated on the same footing as \( \varrho, u. \) EoM: \( \nabla_\alpha \mathcal{T}^\alpha_\beta = 0 \) (+Einstein) supplemented by further equations satisfied by the viscous fluxes. (Moments method.)

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Second-order theory: \( u^\mu \nabla_\mu \pi + \cdots = 0 \) etc.
The Eckart and Landau-Lifshitz theories

Starting from:

\[ T_{\alpha\beta} = (\rho + R)u_{\alpha}u_{\beta} + (p + P)\Pi_{\alpha\beta} + \pi_{\alpha\beta} + Q_{\alpha}u_{\beta} + Q_{\beta}u_{\alpha}. \]

Eckart ('40) and Landau-Lifshitz ('50) (first-order): \( R = 0 \),

\[ \pi_{\alpha\beta} := -2\eta\Pi_{\alpha}^{\mu}\Pi_{\beta}^{\nu}(\nabla_{\mu}u_{\nu} + \nabla_{\nu}u_{\mu} - \frac{2}{3}\nabla_{\lambda}u^{\lambda}g_{\mu\nu}), \quad P := -\zeta\nabla_{\mu}u^{\mu}, \quad (Q_{\alpha} = 0), \]

where \( \eta = \eta(\rho) \), \( \zeta = \zeta(\rho) \) are the coefficients of shear and bulk viscosity.
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In essence:

1. Covariant generalization of Navier-Stokes.
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In essence:

1. Covariant generalization of Navier-Stokes.
2. Entropy production \( \geq 0 \).
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Instability/acausality results apply to large classes of first-order theories. Difficult to construct causal and stable theories of relativistic fluids with viscosity: great deal of work trying to address the issue.
The Israel-Stewart theory

\[ T_{\alpha\beta} = (\rho + R)u_\alpha u_\beta + (p + P)\Pi_{\alpha\beta} + \pi_{\alpha\beta} + Q_\alpha u_\beta + Q_\beta u_\alpha. \]
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EoM: \( R = 0, \nabla_{\alpha}\mathcal{T}^{\alpha}_{\beta} = 0 \) and

\[ \begin{align*}
\tau_P u^\mu \nabla_\mu P + P + \zeta \nabla_\mu u^\mu &= \mathcal{J}^P, \\
\tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu Q_\nu + Q_\alpha &= \mathcal{J}^{Q}_\alpha, \\
\tau_\pi u^\lambda \hat{\Pi}^\mu_\alpha \nabla_\mu \pi_{\lambda\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} &= \mathcal{J}^{\pi}_{\alpha\beta},
\end{align*} \]

where \( \hat{\Pi} \) is the \( u^\perp \) 2-tensor projection onto its symmetric and trace-free part; \( \sigma \) is the \( u^\perp \) trace-free part of \( \nabla u \), \( \tau' s = \tau(\varrho) \) are relaxation times.
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\tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu Q_\nu + Q_\alpha &= \mathcal{J}_\alpha^Q, \\
\tau_\pi u^\lambda \hat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta \sigma_{\alpha\beta} &= \mathcal{J}_{\alpha\beta}^\pi,
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\[ \mathcal{J}' \text{s} = \partial \rho + \partial u + \partial \mathcal{P} + \partial Q + \partial \pi \]
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\[ \tau_P u^\mu \nabla_\mu P + P + \zeta \nabla_\mu u^\mu = J^P, \]

\[ \tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu Q_\nu + Q_\alpha = J^Q_\alpha, \]

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\[ J' s = \partial \rho + \partial u + \partial P + \partial Q + \partial \pi = \text{top order} \]
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\[ \tau_P u^\mu \nabla_\mu P + P + \zeta \nabla_\mu u^\mu = J^P, \]
\[ \tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu Q_\nu + Q_\alpha = J^Q_\alpha, \]
\[ \tau_\pi u^\lambda \hat{\Pi}_{\alpha\beta}^{\mu\nu} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} = J^{\pi}_{\alpha\beta}, \]

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System is highly complex; e.g., \( Q = 0 \), \( 22 \times 22 \) system with non-diagonal principal part.
Features of the Israel-Stewart theory

For the Israel-Stewart theory:

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Limitations of the Israel-Stewart theory

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There exists an open set of smooth initial data for the Israel-Stewart equations for which the corresponding unique smooth solutions to the Cauchy problem break down in finite time. Such data consists of localized (large) perturbations of constant states.
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Proof by contradiction: it does not reveal the nature of the singularity; first breakdown result for Israel-Stewart.
The BDNK theory

The BDNK theory is a first-order theory defined by (D-Bemfica-Noronha, ’18, ’19, ’20; Kovtun, ’19; Hoult-Kovtun, ’20):

\[ T_{\alpha\beta} = (\rho + R)u_{\alpha}u_{\beta} + (p + P)\Pi_{\alpha\beta} + \pi_{\alpha\beta} + Q_{\alpha}u_{\beta} + Q_{\beta}u_{\alpha}, \]

with

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R := \tau_R (u^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu u^\mu),
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P := -\zeta \nabla_\mu u^\mu + \tau_P (u^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu u^\mu),
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Lots of terms: need them to fix the causality and instability problems of Eckart and Landau-Lifshitz. One should let the fundamental principle of causality constrain which terms are allowed in the theory rather than decide the possible terms and then try to establish causality.
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Theorem in fact valid with baryon current and $p = p(\rho, n)$. 

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The BDNK theory has all the good features of the Israel-Stewart theory plus a good local well-posedness theory in Sobolev spaces, which is lacking for Israel-Stewart (applications to neutron star mergers).
Looking ahead

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